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RAPID EXTENSION OF A PENNY-SHAPED CRACK UNDER TORSION

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Abstract—The rapid extension of a penny-shaped crack under torsion is investigated. Both dynamic and quasi-static loading is considered. The wave motion is analyzed through a Green's function technique which leads to an integral equation for the stress field around the crack. Asymptotic expansions for the stress intensity and displacement rate intensity functions which are valid for a small time are obtained for the two types of loading. The propagation of the crack is analyzed through the balance of rates of energy criterion.

1. INTRODUCTION

Under quasi-static loading conditions, effects of inertia on crack propagation are significant if a crack propagates fast enough to generate appreciable wave motions in the medium. Inertia effects are also important if the external loads are rapidly applied and give rise to stress waves which strike the crack and cause fracture.

Analytical investigations of the influence of inertia on crack propagation have been carried out for a number of problems. A survey of elastodynamic fracture studies can be found in a recent paper by Achenbach[1]. Most of the available analytical work is for a plane two dimensional geometry. The diffraction of a plane transient torsional wave by a penny-shaped crack was, however, recently investigated by Sih and Embley[2]. Their approach was primarily numerical, and they did not conisder crack propagation.

It is the purpose of the present study to investigate the rapid propagation of a penny-shaped crack under torsion. Both dynamic and quasi-static loading is considered. The problem is analyzed through a Green's function technique employed by Kostrov[3] and Achenbach[4]. Intensity factors for the shear stress and the particle velocity in the plane of the crack are obtained in the form of asymptotic expansions valid for a small time. For the stationary crack the results agree well with those of [2]. The conditions for crack propagation and the speed of crack propagation are analyzed on the basis of the balance of rates of energies.

An unbounded homogeneous, isotropic, linearly elastic solid is considered. In a system of cylindrical coordinates (r, θ, z), axially symmetric torsional wave motions are defined by a single displacement component $u_\theta\,(r, z, t)$ which is independent of the angular variable θ. The only non-zero stress components are $\tau_{z\theta}$ and $\tau_{r\theta}$ which depend on u_θ as follows

$$\tau_{z\theta} = \mu \frac{\partial u_\theta}{\partial z} \tag{1.1}$$

$$\tau_{r\theta} = \mu \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \tag{1.2}$$

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where μ is the shear modulus. The displacement equation of motion is

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = \frac{\partial^2 u_\theta}{\partial s^2}. \tag{1.3}$$

In equation (1.3)

$$s = ct \tag{1.4}$$

where $c = (\mu/\rho)^{1/2}$ is the velocity of shear waves, and ρ is the mass density.

2. PENNY-SHAPED CRACK OF INCREASING RADIUS

Consider a penny-shaped crack of monotonically increasing radius $a + R(s)$ in a disturbed elastic solid of infinite extent. The crack is located in the $z = 0$ plane of a circular cylindrical coordinate system (r, θ, z) whose origin coincides with the center of the crack (see Fig. 1). The fields of stresses and particle velocities which are examined in this paper can be obtained as superpositions of the solutions to two problems. These are the fields in the absence of a crack and the fields generated by appropriate distributions of tractions on the surfaces of the crack. A method to analyze the latter is outlined in this section.

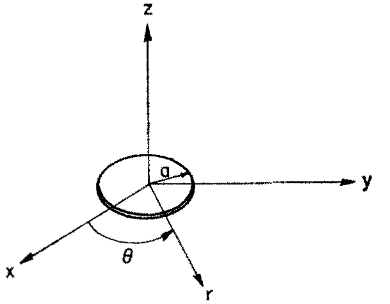

Fig. 1. Geometry of the penny-shaped crack.

Suppose that the surfaces of the crack are subjected to shear tractions of the type

$$\tau_{z\theta}(r, s) = -\tau(r, s)H(s) \quad \text{for} \quad r < a + R(s), \qquad z = \pm 0. \tag{2.1}$$

Here $H(s)$ is the Heaviside Step function. It is assumed that the original radius of the crack is $r = a$, and that the crack begins to propagate at a time $s = s_f$, i.e.

$$R(s) = 0 \quad \text{for} \quad s \leq s_f. \tag{2.2}$$

It is also assumed that

$$0 \leq \mathrm{d}R/\mathrm{d}s < 1. \tag{2.3}$$

Since the surface tractions are equally applied to the two sides of the crack, it follows from symmetry considerations that the rotational displacement $u_\theta(r, z, s)$ vanishes in the plane of the crack outside of its boundaries. Thus,

$$u_\theta = 0 \quad \text{for} \quad r \geq a + R(s), \qquad z = 0. \tag{2.4}$$

The surface tractions are applied at time $t = 0$, which implies the following initial conditions

$$u_\theta(r, z, s) = \frac{\partial}{\partial s} u_\theta(r, z, s) \equiv 0 \quad \text{for} \quad s < 0. \tag{2.5}$$

A problem of the type defined by equation (1.1)–(1.3) and (2.1)–(2.5) can be solved by means of a Green's function approach.

The response of a half-space to a distribution of surface tractions $\tau_{z\theta}(r, s)$ which is independent of θ and of arbitrary dependence on r and s may be obtained by first finding the response to a ring load, and then employing linear superposition. The traction boundary condition for a unit ring load applied at $r = \bar{r}$ at time $s = \bar{s}$ may be expressed as

$$\tau_{z\theta}(r, \theta, s) = \frac{1}{2\pi r} \delta(r - \bar{r})\, \delta(s - \bar{s}) \quad \text{at} \quad z = 0, \tag{2.6}$$

where $\delta(\cdot)$ is the Dirac delta function. To determine the response to surface tractions of the form (2.6), a double transform involving a one-sided Laplace transform on s and a Hankel transform of order one on r, (see [5]) will be employed, i.e.

$$\hat{u}_\theta(q, z, p) = \int_0^\infty \int_0^\infty u_\theta(r, z, s) e^{-ps} J_1(qr) r \, dr \, ds, \tag{2.7}$$

where p and q are the Laplace and Hankel transform variables, respectively. After transforming equation (1.1), (1.3) and (2.6), the transformed displacement is obtained as

$$\hat{u}_\theta(q, z, p) = -\frac{J_1(q\bar{r})}{2\pi\mu(q^2 + p^2)^{1/2}} \exp - [\bar{s}p + (q^2 + p^2)^{1/2} z]. \tag{2.8}$$

The transform inversion is readily performed by employing the appropriate formulae in Erdélyi *et al.*[6, 7]. The result is the Green's function

$$G(r, z, s; \bar{r}, \bar{s}) = -\frac{1}{2\mu\pi^2} \frac{1}{r\bar{r}} \frac{[r^2 + \bar{r}^2 + z^2 - (s - \bar{s})^2]}{\{4r^2\bar{r}^2 - [r^2 + \bar{r}^2 + z^2 - (s - \bar{s})^2]^2\}^{1/2}}$$
$$\times H[(s - \bar{s})^2 - (r - \bar{r})^2 - z^2] H[(r + \bar{r})^2 + z^2 - (s - \bar{s})^2]. \tag{2.9}$$

By virtue of linear superposition the displacement at the surface of the half-space due to an axisymmetric surface traction $\tau_{z\theta}(r, s)$ may now be expressed as

$$u_\theta(r, s) = 2\pi \int_0^\infty \int_0^\infty \tau_{z\theta}(\bar{r}, \bar{s}) G(r, 0, s; \bar{r}, \bar{s}) \bar{r} \, d\bar{r} \, d\bar{s}, \tag{2.10}$$

where $G(r, 0, s; \bar{r}, \bar{s})$ follows from equation (2.9) by setting $z = 0$. The Heaviside step functions in equation (2.9) define the actual region of integration in the $\bar{r} - \bar{s}$ plane.

Let us now return to the problem defined by equation (2.1)–(2.5), and let us consider the half-plane $z \geq 0$, with the boundary conditions (2.1) and (2.4). Since the displacement is known to vanish outside of the area encompassed by $r = a + R(s)$, equation (2.10) may be employed as an integral equation for the unknown shear stress $\tau_{z\theta}$ in the region $r > a + R(s)$. Considering a point $r > a + R(s)$, $s > r - a$ we find

$$\iint_{S_1} \tau(\bar{r}, \bar{s}) g(r, s; \bar{r}, \bar{s}) \, d\bar{r} \, d\bar{s} + \iint_{S_2} \tau_{z\theta}(\bar{r}, \bar{s}) g(r, s; \bar{r}, \bar{s}) \, d\bar{r} \, d\bar{s} = 0, \tag{2.11}$$

where $\tau(\bar{r}, \bar{s})$ is the applied surface traction, see equation (2.1), and

$$g(r, s; \bar{r}, \bar{s}) = -\frac{1}{r}\frac{[(r+\bar{r})^2 - (s-\bar{s})^2] - [(s-\bar{s})^2 - (r-\bar{r})^2]}{[(r+\bar{r})^2 - (s-\bar{s})^2]^{1/2}[(s-\bar{s})^2 - (r-\bar{r})^2]^{1/2}}. \tag{2.12}$$

It is not difficult to see that S_1 and S_2 are the trapezoidal regions which are indicated as shaded regions in Fig. 2.

To solve the integral equation (2.11) it is convenient to introduce the change of coordinates

$$\xi = [s - (r-a)]/\sqrt{2}, \qquad \eta = [s + (r-a)]/\sqrt{2}. \tag{2.13a,b}$$

The function $g(r, s; \bar{r}, \bar{s})$ becomes

$$g(r, s; \bar{r}, \bar{s}) = h(\xi, \eta; \bar{\xi}, \bar{\eta}), \tag{2.14}$$

where

$$h(\xi, \eta; \bar{\xi}, \bar{\eta}) = \frac{-\sqrt{2}}{a\sqrt{2} + (\eta - \xi)}\frac{[a\sqrt{2} + (\bar{\eta} - \xi)][a\sqrt{2} + (\eta - \bar{\xi})] - (\xi - \bar{\xi})(\eta - \bar{\eta})}{\{[a\sqrt{2} + (\bar{\eta} - \xi)][a\sqrt{2} + (\eta - \bar{\xi})](\xi - \bar{\xi})(\eta - \bar{\eta})\}^{1/2}}. \tag{2.15}$$

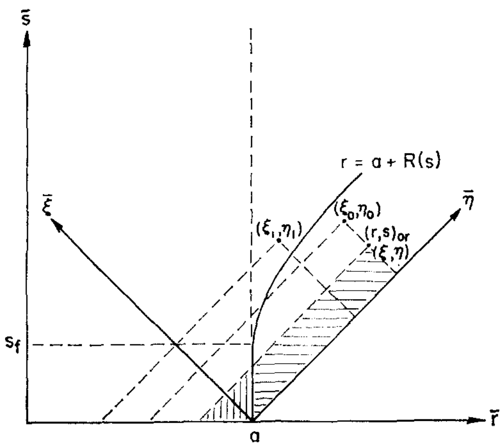

Fig. 2. Regions in the $\bar{r}$–$\bar{s}$ plane.

For a point ξ, η corresponding to $r > a$, $r - a < s < r - a + s_f$ the integral equation assumes the form

$$\int_0^{\xi} d\bar{\xi} \int_{-\bar{\xi}}^{\bar{\xi}} \tau(\bar{\xi}, \bar{\eta}) h(\xi, \eta; \bar{\xi}, \bar{\eta})\, d\bar{\eta} + \int_0^{\xi} d\bar{\xi} \int_{\bar{\xi}}^{\eta} \tau_{z\theta}(\bar{\xi}, \bar{\eta}) h(\xi, \eta; \bar{\xi}, \bar{\eta})\, d\bar{\eta} = 0. \tag{2.16}$$

In the next sections we will consider the case $r > a + R(s)$, $r - a < s$.

In its present form there appears little hope of obtaining an analytical solution to equation (2.16). Consequently a solution is sought which is valid for small time; i.e. it is assumed that s/a is small. If s/a is a small number, ξ/a must also be small; and $h(\xi, \eta; \bar{\xi}, \bar{\eta})$ may be expanded in a series of powers of (ξ/a) as

$$h(\xi, \eta; \bar{\xi}, \bar{\eta}) = \frac{-1}{(\xi - \bar{\xi})^{1/2}(\eta - \bar{\eta})^{1/2}}\left\{1 + \frac{1}{2\sqrt{2}}\left[\frac{\bar{\eta} - \bar{\xi}}{\xi} - \frac{\eta - \xi}{\xi}\right]\left(\frac{\xi}{a}\right) + O\left[\left(\frac{\xi}{a}\right)^2\right]\right\}. \tag{2.17}$$

The unknown shear stress $\tau_{z\theta}(\xi, \eta)$ is now also expanded in the form

$$\tau_{z\theta}(\xi, \eta) = \tau_0(\xi, \eta) + \tau_1(\xi, \eta)(\xi/a) + O[(\xi/a)^2]. \tag{2.18}$$

Upon substituting equation (2.17) and (2.18) into the integral equation (2.16), and letting the coefficients of powers of (ξ/a) vanish independently, we obtain integral equations for $\tau_0(\xi, \eta)$ and $\tau_1(\xi, \eta)$. These integral equations can be solved, as will be shown for specific examples in the next sections.

3. DYNAMIC LOADING

In this section we investigate the stress distribution near the edge of a penny-shaped crack in an elastic body for the case that crack propagation is generated when a transient stress wave is diffracted by the crack. The incident stress wave is of the form

$$\tau_{z\theta} = (r\tau/a)H(s - z), \tag{3.1}$$

where τ is a constant. If the crack did not exist, the wave motion would cause a stress distribution of the form

$$\tau_{z\theta} = (r\tau/a)H(s) \tag{3.2}$$

to appear in the plane $z = 0$. The solution of the diffraction problem caused by the presence of the crack may be obtained by superimposing upon the stress wave defined by equation (3.1) the stress distributions that are generated by the application of a stress equal and opposite to equation (3.2) over the area encompassed by the two sides of the crack. The effect of this superposition is to render the surface of the crack free of tractions. The superimposed problem is defined by equations (1.1)–(1.3) and equations (2.2)–(2.5), while equation (2.1) is replaced by

$$\tau_{z\theta} = -(r\tau/a)H(s) \quad \text{for} \quad r < a + R(s), \qquad z = \pm 0. \tag{3.3}$$

Stationary crack

The stress field in the plane of the crack after it has been struck by the incident wave but before it begins to propagate will be determined first. This will permit the results of the present study to be compared with those of Sih and Embley[2]. By introducing equations (2.13 a,b) in (3.3), and substituting the results together with equation (2.17) and (2.18) into the integral equation (2.16), we find

$$I_1 + \left(\frac{\xi}{a}\right)\left[\frac{\eta - \xi}{2\xi\sqrt{2}} I_1 + \frac{1}{\xi} I_2\right] + O\left[\left(\frac{\xi}{a}\right)^2\right] = 0, \tag{3.4}$$

where

$$I_1 = \int_0^\xi \frac{d\bar{\xi}}{(\xi - \bar{\xi})^{1/2}}\left[-\tau \int_{-\bar{\xi}}^{\bar{\xi}} \frac{d\bar{\eta}}{(\eta - \bar{\eta})^{1/2}} + \int_{\bar{\xi}}^{\eta} \frac{\tau_0(\bar{\xi}, \bar{\eta})\, d\bar{\eta}}{(\eta - \bar{\eta})^{1/2}}\right], \tag{3.5}$$

$$I_2 = \int_0^\xi \frac{d\bar{\xi}}{(\xi - \bar{\xi})^{1/2}}\left\{-\frac{3\tau}{2\sqrt{2}} \int_{-\bar{\xi}}^{\bar{\xi}} \frac{(\bar{\eta} - \bar{\xi})\, d\bar{\eta}}{(\eta - \bar{\eta})^{1/2}} + \int_{\bar{\xi}}^{\eta}\left[\bar{\xi}\tau_1(\bar{\xi}, \bar{\eta}) + \frac{\bar{\eta} - \bar{\xi}}{2\sqrt{2}}\tau_0(\bar{\xi}, \bar{\eta})\right]\frac{d\bar{\eta}}{(\eta - \bar{\eta})^{1/2}}\right\}. \tag{3.6}$$

Setting $I_1 = 0$ and $I_2 = 0$ yields two integral equations which may be solved in turn for the first two terms in the expansion for $\tau_{z\theta}(\xi, \eta)$. The first equation is

$$\int_{\xi}^{\eta} \frac{\tau_0(\xi, \bar{\eta})\, d\bar{\eta}}{(\eta - \eta)^{1/2}} = \tau \int_{-\xi}^{\xi} \frac{d\bar{\eta}}{(\eta - \bar{\eta})^{1/2}}. \tag{3.7}$$

This may be recognized as an Abel integral equation whose solution is

$$\tau_0(\xi, \eta) = \frac{\tau}{\pi} \left\{ \frac{2(2\xi)^{1/2}}{(\eta - \xi)^{1/2}} - 2 \tan^{-1} \left[\frac{2\xi}{(\eta - \xi)} \right]^{1/2} \right\}. \tag{3.8}$$

The integral equation for $\tau_1(\xi, \eta)$ follows from $I_2 = 0$. The solution to that equation is

$$\tau_1(\xi, \eta) = \frac{\tau}{\pi} \{ -2\xi^{1/2}/(\eta - \xi)^{1/2} + 2\xi^{1/2}(\eta - \xi)^{1/2} - (\eta - \xi)\sqrt{2} \tan^{-1}[2\xi/(\eta - \xi)]^{1/2} \}. \tag{3.9}$$

Of primary interest in the expression for $\tau_{z\theta}$ is the singular term near the edge of the crack. Returning to the r–s coordinate system, the expansion for the shear stress in the plane of the crack in the vicinity of the edge is

$$\tau_{z\theta}(r, s) = T(s)/(r - a)^{1/2} \tag{3.10}$$

where the stress-intensity function is obtained as

$$T(s) = (2\tau/\pi)s^{1/2}\{1 - \tfrac{1}{2}(s/a) + O[(s/a)^2]\}. \tag{3.11}$$

The ratio of this dynamic stress intensity function to the corresponding static stress intensity function

$$T_{st} = 4\tau a^{1/2}/3\pi\sqrt{2} \tag{3.12}$$

is plotted in Fig. 3 as a function of s/a for $s/a < 1$. The dashed line represents the values

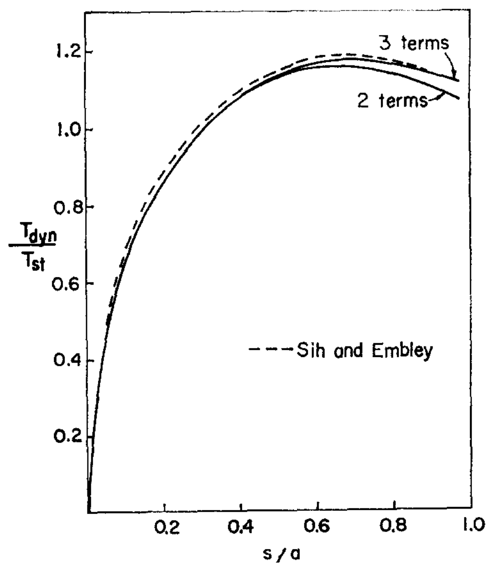

Fig. 3. Variation of stress intensity function with time.

obtained by Sih and Embley[2]. The lower solid curve represents the values obtained by employing the first two terms of the expansion of the stress intensity function given by equation (3.11). The coefficient of the third term in this expansion was also calculated, and its magnitude was found to be only 1/20 the magnitude of the coefficient of the second term. The upper solid curve represents the stress intensity function given by a three term expansion. It is seen that the additional term renders a slight improvement in accuracy only for values of s/a beyond the peak in the curve. It is difficult to give a justification for using the expansion of the stress intensity function for such relatively large values of s/a. Equation (3.11) is an asymptotic expansion, and there is no proof that this expansion converges as s/a approaches unity.

Figure 3 shows a dynamic overshoot of about 20 per cent of the dynamic stress intensity factor over the corresponding static stress intensity factor. In view of this dynamic amplification of the stress level it is conceivable that fracture does not occur when the loads are gradually applied, but that the material does indeed fracture when the loads are rapidly applied and give rise to waves which strike the crack.

Propagating crack

Now the stress field in the plane of the crack will be considered after the crack has started to propagate, i.e. for $s > s_f$. Consider the point (ξ_0, η_0) in Fig. 2. The integral equation corresponding to equation (2.11) is

$$
\begin{aligned}
&-\tau \int_0^{\xi_f} \mathrm{d}\bar{\xi} \int_{-\bar{\xi}}^{\bar{\xi}} [1 + (\bar{\eta} - \bar{\xi})/a\sqrt{2}] h(\xi_0, \eta_0; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta} \\
&+ \int_0^{\xi_f} \mathrm{d}\bar{\xi} \int_{\bar{\xi}}^{\eta_0} \tau_{z\theta}(\bar{\xi}, \bar{\eta}) h(\xi_0, \eta_0; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta} \\
&-\tau \int_{\xi_f}^{\xi_0} \mathrm{d}\bar{\xi} \int_{-\bar{\xi}}^{N(\bar{\xi})} [1 + (\bar{\eta} - \bar{\xi})/a\sqrt{2}] h(\xi_0, \eta_0; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta} \\
&+ \int_{\xi_f}^{\xi_0} \mathrm{d}\bar{\xi} \int_{N(\bar{\xi})}^{\eta_0} \tau_{z\theta}(\bar{\xi}, \bar{\eta}) h(\xi_0, \eta_0; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta} = 0 \qquad (3.13)
\end{aligned}
$$

where $\xi_f = s_f/\sqrt{2}$, and $N(\xi)$ is the solution of

$$[N(\xi) - \xi]/\sqrt{2} = R\{[N(\xi) + \xi]/\sqrt{2}\}. \qquad (3.14)$$

When the expansions (2.17) and (2.18) are introduced into equation (3.13), terms of equal order in (ξ/a) are collected, and equations (3.8) and (3.9) are employed, the remaining integral equations yield the stress in the vicinity of the edge of the crack as

$$\tau_{z\theta}(\xi, \eta) = \frac{2\tau}{\pi} \frac{[\xi + N(\xi)]^{1/2}}{[\eta - N(\xi)]^{1/2}} \left\{1 - \frac{3\xi - N(\xi)}{2\xi\sqrt{2}} \left(\frac{\xi}{a}\right) + O\left[\left(\frac{\xi}{a}\right)^2\right]\right\} + O(1). \qquad (3.15)$$

It is noted that the new subscripts have been dropped. Using a limiting procedure similar to the one used in [4], the stress may be expressed in the form

$$\tau_{z\theta}(r, s) = T(s)/[r - a - R(s)]^{1/2} + O(1) \qquad (3.16)$$

where

$$T(s) = \frac{2\tau s^{1/2}}{\pi} \left(1 - \frac{dR}{ds}\right)^{1/2} \left\{1 - \frac{s - 2R}{2s} \left(\frac{s}{a}\right) + O\left[\left(\frac{s}{a}\right)^2\right]\right\}. \qquad (3.17)$$

It is noteworthy that $T(s)$ generally decreases as $\mathrm{d}R/\mathrm{d}s$ increases. The stress intensity function vanishes for $\mathrm{d}R/\mathrm{d}s = 1$, which corresponds to a crack propagating with the velocity of transverse waves, c_T.

The displacement on the surface of the crack will now be obtained for the point (ξ_1, η_1), shown in Fig. 2. It may be expressed as

$$\begin{aligned}2\mu\pi u_\theta(\xi_1, \eta_1) = &-\tau \int_0^{\xi_f} \mathrm{d}\bar{\xi} \int_{-\bar{\xi}}^{\bar{\xi}} [1 + (\bar{\eta} - \bar{\xi})/a\sqrt{2}] h(\xi_1, \eta_1; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta} \\ &+ \int_0^{\xi_f} \mathrm{d}\bar{\xi} \int_{\bar{\xi}}^{\eta_1} \tau_{z\theta}(\bar{\xi}, \bar{\eta}) h(\xi_1, \eta_1; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta} \\ &- \tau \int_{\xi_f}^{K(\eta_1)} \mathrm{d}\bar{\xi} \int_{-\bar{\xi}}^{N(\bar{\xi})} [1 + (\bar{\eta} + \bar{\xi})/a\sqrt{2}] h(\xi_1, \eta_1; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta} \\ &+ \int_{\xi_f}^{K(\eta_1)} \mathrm{d}\bar{\xi} \int_{N(\bar{\xi})}^{\eta_1} \tau_{z\theta}(\bar{\xi}, \bar{\eta}) h(\xi_1, \eta_1; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta} \\ &+ \tau \int_{K(\eta_1)}^{\xi_1} \mathrm{d}\bar{\xi} \int_{-\bar{\xi}}^{\eta_1} [1 + (\bar{\eta} - \bar{\xi})/a\sqrt{2}] h(\xi_1, \eta_1; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\eta}, \end{aligned} \tag{3.18}$$

where

$$[\eta - K(\eta)]/\sqrt{2} = R\{[\eta + K(\eta)]/\sqrt{2}\}. \tag{3.19}$$

After introducing the expansions (2.17) and (2.18), the first and the third integrals in equation (3.18) cancel the second and the fourth integrals, respectively. From the fifth integral the displacement in the vicinity of the edge of the crack is obtained as

$$u_\theta(\xi, \eta) = \frac{2\pi\sqrt{2}}{\mu\pi} [\eta + K(\eta)]^{1/2} [\xi - K(\eta)]^{1/2} \left\{1 - \frac{3K(\eta) - \eta}{2\xi\sqrt{2}} \left(\frac{\xi}{a}\right) + O\left[\left(\frac{\xi}{a}\right)^2\right]\right\} + O[\xi - K(\eta)]. \tag{3.20}$$

Expressing this function in the r, s-coordinates and differentiating with respect to s, we obtain

$$\frac{\partial}{\partial s} u_\theta(r, s) = \dot{U}_\theta(s)/[a + R(s) - r]^{1/2} + O(1) \tag{3.21}$$

where the displacement-rate intensity function may be written as

$$\dot{U}(s) = \frac{2\tau s^{1/2}}{\mu\pi} \left(1 + \frac{dR}{\mathrm{d}s}\right)^{-1/2} \frac{dR}{\mathrm{d}s} \left\{1 - \frac{s - 2R}{2s} \left(\frac{s}{a}\right) + O\left[\left(\frac{s}{a}\right)^2\right]\right\}. \tag{3.22}$$

This intensity function vanishes for $\mathrm{d}R/\mathrm{d}s = 0$, and generally increases as $\mathrm{d}R/\mathrm{d}s$ increases.

It should be noted that within a restricted time domain the results of this section apply to the geometry of a circular cylinder of radius b which contains a penny-shaped crack of radius a centered at the axis of the cylinder and placed normal to the axis. When an end-section of the cylinder is subjected to a uniform rotation which increases linearly with time, the resulting stress wave is of the form (3.1). In the time domain $\tau < 2(b - a)/c_T$ the stress distribution near $r = a$ is just the same as for an unbounded body containing a crack and subjected to a wave motion of the form (3.1). In an analogous manner the technique of this paper can also be used for the case that the cylinder has a saw-cut type flaw along a circumference.

4. STATIC LOADING

In this section the elastic body containing the penny-shaped crack shown in Fig. 1 is assumed to be loaded quasi-statically to a critical level. The crack begins to propagate at time $s = 0$, while the externally applied loads are maintained at a constant level. It is assumed that crack propagation occurs so rapidly that dynamic effects cannot be ignored.

Again a superposition technique will be employed to find the stress distribution around the propagating crack. The static stress distribution in the plane of a stationary penny-shaped crack in an infinite body under a particular kind of torsion may be found in the literature, e.g. [8]. The stress in the plane $z = 0$ is

$$\tau_{z\theta} = 0, \qquad r < a \tag{4.1}$$

$$\tau_{z\theta}/\tau = f(r) = \frac{(r^2 - a^2)^{1/2}}{r} + \frac{a^2}{2r(r^2 - a^2)^{1/2}}, \qquad r > a \tag{4.2}$$

where τ, which is the shear stress at $(r^2 + z^2)^{1/2} \to \infty$, is a constant. An equal and opposite stress is imposed over the fracture surface $a < r < R(s)$ to render it free of traction. The dynamic stresses generated by the fracture process are thus obtained by finding a solution in the half space $z \geq 0$ which satisfies equations (1.3), (2.3) and (2.5), and the following boundary conditions at $z = 0$

$$\tau_{z\theta} = 0, \qquad r < a \tag{4.3}$$

$$\tau_{z\theta} = -\tau f(r), \qquad a < r < a + R(s) \tag{4.4}$$

$$u_\theta = 0, \qquad r \geq a + R(s). \tag{4.5}$$

In the r–s and ξ–η plane the situation is as shown in Fig. 2, except that s_f equals zero. Considering a point (ξ, η) ahead of the propagating edge of the crack, where the displacement is zero, the integral equation for the unknown shear stress $\tau_{z\theta}$ is obtained as

$$\tau \int_0^\xi \mathrm{d}\bar{\xi} \int_{\bar{\xi}}^{N(\bar{\xi})} f(\bar{\xi}, \bar{\eta}) h(\xi, \eta; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\xi}\, \mathrm{d}\bar{\eta} + \int_0^\xi \mathrm{d}\bar{\xi} \int_{N(\bar{\xi})}^{\eta} \tau_{z\theta}(\bar{\xi}, \bar{\eta}) h(\xi, \eta; \bar{\xi}, \bar{\eta})\, \mathrm{d}\bar{\xi}\, \mathrm{d}\bar{\eta} = 0, \tag{4.6}$$

where $f(\bar{\xi}, \bar{\eta})$ follows from $f(r)$, see equation (4.2), as

$$f(\xi, \eta) = -\frac{(\eta - \xi)^{1/2}[(\eta - \xi) + 2a\sqrt{2}]^{1/2} + a^2(\eta - \xi)^{-1/2}[(\eta - \xi) + 2a\sqrt{2}]^{-1/2}}{(\eta - \xi) + a\sqrt{2}}. \tag{4.7}$$

A small time solution will be sought. Assuming that ξ/a is small, $f(\xi, \eta)$ may be expressed as

$$f(\xi, \eta) = -\frac{1}{2(2)^{1/4}} \left(\frac{\xi}{\eta - \xi}\right)^{1/2} \left(\frac{a}{\xi}\right)^{1/2} \left\{1 + \frac{11}{4\sqrt{2}} \frac{\eta - \xi}{\xi} \left(\frac{\xi}{a}\right) + O\left[\left(\frac{\xi}{a}\right)^2\right]\right\}. \tag{4.8}$$

The unknown shear stress is assumed to be of the form

$$\tau_{z\theta}(\xi, \eta) = \left(\frac{a}{\xi}\right)^{1/2} \left\{\tau_0(\xi, \eta) + \tau_1(\xi, \eta)\left(\frac{\xi}{a}\right) + O\left[\left(\frac{\xi}{a}\right)^2\right]\right\}. \tag{49}$$

Proceeding as before, these expressions may be substituted into the integral equation, and the coefficients of each power of (ξ/a) set equal to zero. This yields integral equations for the terms of the expansion of $\tau_{z\theta}(\xi, \eta)$. After solving these integral equations, the stress in the vicinity of the moving edge of the crack may be expressed as

$$\tau_{z\theta}(\xi, \eta) = \frac{\tau}{2(2)^{1/4}}\left(\frac{a}{\eta - N(\xi)}\right)^{1/2}\left\{1 + \frac{9}{8\sqrt{2}}\frac{N(\xi) - \xi}{\xi}\left(\frac{\xi}{a}\right) + O\left[\left(\frac{\xi}{a}\right)^2\right]\right\} + O\{[\eta - N(\xi)]^{1/2}\}. \tag{4.10}$$

Writing this expression in the r, s-coordinates, the stress intensity function is found as

$$T(s) = \frac{\tau}{2}\left(\frac{a}{2}\right)^{1/2}\left(1 - \frac{dR}{ds}\right)^{1/2}\left\{1 + \frac{9}{8}\frac{R}{s}\left(\frac{s}{a}\right) + O\left[\left(\frac{s}{a}\right)^2\right]\right\}. \tag{4.11}$$

The displacement inside the crack tip may be obtained in the usual way, and the displacement-rate intensity function may be expressed as

$$\dot{U}(s) = \frac{1}{2}\frac{\tau}{\mu}\left(\frac{a}{2}\right)^{1/2}\left(1 + \frac{dR}{ds}\right)^{-1/2}\frac{dR}{ds}\left\{1 + \frac{9}{8}\frac{R}{s}\left(\frac{s}{a}\right) + O\left[\left(\frac{s}{a}\right)^2\right]\right\}. \tag{4.12}$$

The dynamic stress intensity factor decreases with $\mathrm{d}R/\mathrm{d}s$, see equation (4.11). The corresponding quasi-static stress intensity factor, which follows easily from equation (4.2), does not depend explicitly on $\mathrm{d}R/\mathrm{d}s$.

5. DISCUSSION

In the preceding sections we have computed the magnitudes of stress intensity factors and displacement-rate intensity factors. It remains to relate these magnitudes to the process of fracture, particularly to the onset of fracture and to the speed of crack propagation, by means of a necessary condition for fracture. For brittle elastic materials the appropriate criterion is provided by the balance of rates of energies. This criterion not only serves to determine the condition for onset of fracture; it generally also provides an equation for the computation of the speed of crack propagation. In its original form the criterion was advanced by Griffith[9] as an energy criterion for the onset of quasi-static fracture.

The balance of rates of energies was discussed in considerable detail in [1]. As the material fractures, energy is dissipated. The fracture process can only occur if there is sufficient energy available. It is shown in [1] that the following equality must hold during the fraction process

$$\frac{\mathrm{d}D}{\mathrm{d}s} = -\frac{\mathrm{d}E}{\mathrm{d}s}. \tag{5.1}$$

Here $\mathrm{d}D/\mathrm{d}s$ represents the rate of uptake of surface energy, while $\mathrm{d}E/\mathrm{d}s$ is the rate of work of the cohesive tractions in the plane of the crack as the crack propagates. The latter term may be expressed in terms of the particle velocity just behind the propagating crack tip and the shear stress just in front of the crack tip as

$$-\frac{dE}{ds} = 4\pi\int_{a+R(s)-\varepsilon}^{a+R(s)+\varepsilon}\tau_{z\theta}(r, s)\frac{\partial u_\theta(r, s)}{\partial s}\, r\,\mathrm{d}r \tag{5.2}$$

where ε is a small positive number. Although the shear stress vanishes behind the crack tip and the particle velocity vanishes ahead of the crack tip, the integral in equation (5.2) is still non-zero due to the singularities in these terms at the crack tip. Making use of the relation

$$\frac{H(v)}{v^{1/2}}\cdot\frac{H(-v)}{(-v)^{1/2}} = \frac{\pi}{2}\,\delta(v) \tag{5.3}$$

where $\delta(v)$ is the Dirac delta function (this relation was derived in [1]), dE/ds may be expressed as

$$\frac{dE}{ds} = 2\pi^2 T(s)\dot{U}(s)[a + R(s)], \tag{5.4}$$

where $T(s)$ and $\dot{U}(s)$ are the stress intensity and displacement-rate intensity functions, respectively.

An expression for the dissipation term may be written as

$$\frac{dD}{ds} = 4\pi \frac{d}{ds}\int_a^{R(s)+a} \gamma_F \, r \, dr, \tag{5.5}$$

where γ_F is the amount of energy needed to create a unit area of fracture surface and is called the specific fracture energy of the solid. In the case of brittle fracture, the specific fracture energy has usually been assumed to be a material property, constant with respect to time and crack velocity. Therefore, equation (5.5) becomes

$$\frac{dD}{ds} = 4\pi\gamma_F(R + a)\frac{dR}{ds}. \tag{5.6}$$

The balance equation (5.1) may now be written as

$$\pi T(s)\dot{U}(s) = 2\gamma_F \frac{dR}{ds}. \tag{5.7}$$

The balance of rate of energy equation (5.7) must be satisfied at all times for fracture to occur. The case of dynamic loading will be considered first. Substituting the expressions for $T(s)$ and $\dot{U}(s)$ from equations (3.17) and (3.22) and assuming dR/ds is not identically zero, the balance equation becomes

$$\frac{2\tau^2 s}{\pi\mu}\left(1 - \frac{dR}{ds}\right)^{1/2}\left(1 + \frac{dR}{ds}\right)^{-1/2}\left\{1 - \frac{(s - 2R)}{s}\left(\frac{s}{a}\right) + O\left[\left(\frac{s}{a}\right)^2\right]\right\} = \gamma_F. \tag{5.8}$$

The crack will start to propagate at time $s = s_f$. It is apparent that $s_f = 0$ will not satisfy the balance equation; therefore, the crack cannot propagate immediately upon being struck by the torsional wave. This conclusion is a result of the assumptions made on the behavior of γ_F and the type of loading imposed on the body. The minimum value of s_f may be computed from equation (5.8) by setting R and dR/ds equal to zero which yields

$$s_f = \frac{\pi\mu\gamma_F}{2\tau^2}\left\{1 + \frac{\pi\mu\gamma_F}{2\tau^2 a} + O\left[\left(\frac{\pi\mu\gamma_F}{2\tau^2 a}\right)^2\right]\right\}. \tag{5.9}$$

The function $R(s)$ which specifies the location of the crack tip may be obtained by solving (5.8) as an ordinary differential equation. $R(s)$ is found to be

$$\begin{aligned} R(s) = {} & [s - (2/\alpha)\tan^{-1}(\alpha s) + C_0] + [(s\alpha^2)^{-1}\{2\log(\alpha^2 s^2 + 1) \\ & + 4C_0\alpha\tan^{-1}(\alpha s) + 4[\tan^{-1}(\alpha s)]^2\} - [s(\alpha^2 s^2 + 1)]^{-1}[2s^2 \\ & + 4C_0 s - 8(s/\alpha)\tan^{-1}(\alpha s) + (4/\alpha^2)] + (C_1/s)](s/a) \\ & + O[(s/\alpha)^2], \end{aligned} \tag{5.10}$$

where

$$\alpha = \frac{2\tau^2}{\pi\mu\gamma_F}, \tag{5.11}$$

and C_0 and C_1 are constants whose values depend on the initial conditions for $R(s)$ at $s = s_f$.

Fracture under quasi-static loading may be treated in a similar manner. The values of $T(s)$ and $\dot{U}(s)$ from equation (4.11) and (4.12) are substituted into the balance equation yielding

$$\frac{\pi\tau^2 a}{16\mu}\left(1-\frac{\mathrm{d}R}{\mathrm{d}s}\right)^{1/2}\left(1+\frac{\mathrm{d}R}{\mathrm{d}s}\right)^{-1/2}\left\{1+\frac{9}{4}\frac{R}{s}\left(\frac{s}{a}\right)+O\left[\left(\frac{s}{a}\right)^2\right]\right\}=\gamma_F. \tag{5.12}$$

The minimum stress above which the fracture process will begin may be obtained by setting s, R, and $\mathrm{d}R/\mathrm{d}s$ equal to zero in this equation. This critical stress is found to be

$$\tau_{cr}=[16\gamma_F\,\mu/(\pi a)]^{1/2} \tag{5.13}$$

which agrees with the static result given in[8]. If the stress is below this level, the balance equation will not be satisfied; and fracture cannot occur.

Again the function $R(s)$ may be obtained from equation (5.12) as

$$R(s)=\left(\frac{\beta^2-1}{\beta^2+1}\right)s\left\{1+\frac{9}{2}\frac{\beta^2}{(\beta^2+1)^2}\left(\frac{s}{a}\right)+O\left[\left(\frac{s}{a}\right)^2\right]\right\} \tag{5.14}$$

where

$$\beta=\left(\frac{\tau}{\tau_{cr}}\right)^2. \tag{5.15}$$

In conclusion, it may be stated that a penny-shaped crack which is struck by an incident torsional wave may begin to propagate a short time after it is struck. This is consistent with the results of[4] where an analogous two-dimensional antiplane problem was investigated. Also, a crack in a body which is loaded quasi-statically may begin to propagate if the load exceeds a certain critical value. In the early stages this crack will propagate with constantly increasing velocity.

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Абстракт — Исследуется быстрое распространение подверженной кручению, копейкообразной трещины. Учитываются как динамическая, так и квазистатическая нагрузки. Рассматривается волновое движение посредством метода функции Грина, что приводит к интегральному уравнению для поля напряжений вокруг трещины. Для двух типов нагрузок получаются асимптотические разложения функций интенсивности напряжений и интенсивности скорости деформации, имеющие силу для небольших периодов времени. На основе баланса скоростей для критерия энергии, дается анализ распространения трещины.